Sensometrics Meeting
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Ratios of Normal Variables in Superiority Claims

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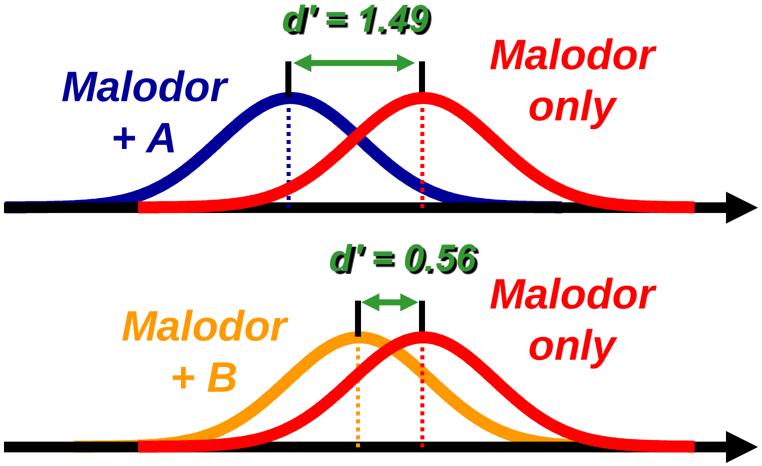
Ratio Statement Examples

Compared to a competitor…

- Carpet treatment **reduces** malodor **five times better**
- Tooth whitening treatment **is twice as effective**
- Air freshener **lasts 20% longer**
- Cleaning product **performs "up to 30%" better**

Example: Malodor Reduction Ratio

- For each product:
  - Application with active ingredient to malodor
  - Application without active ingredient to malodor
- Rating based on malodor intensity in odor test chambers
- Use d′ values to work with differences on an interval scale



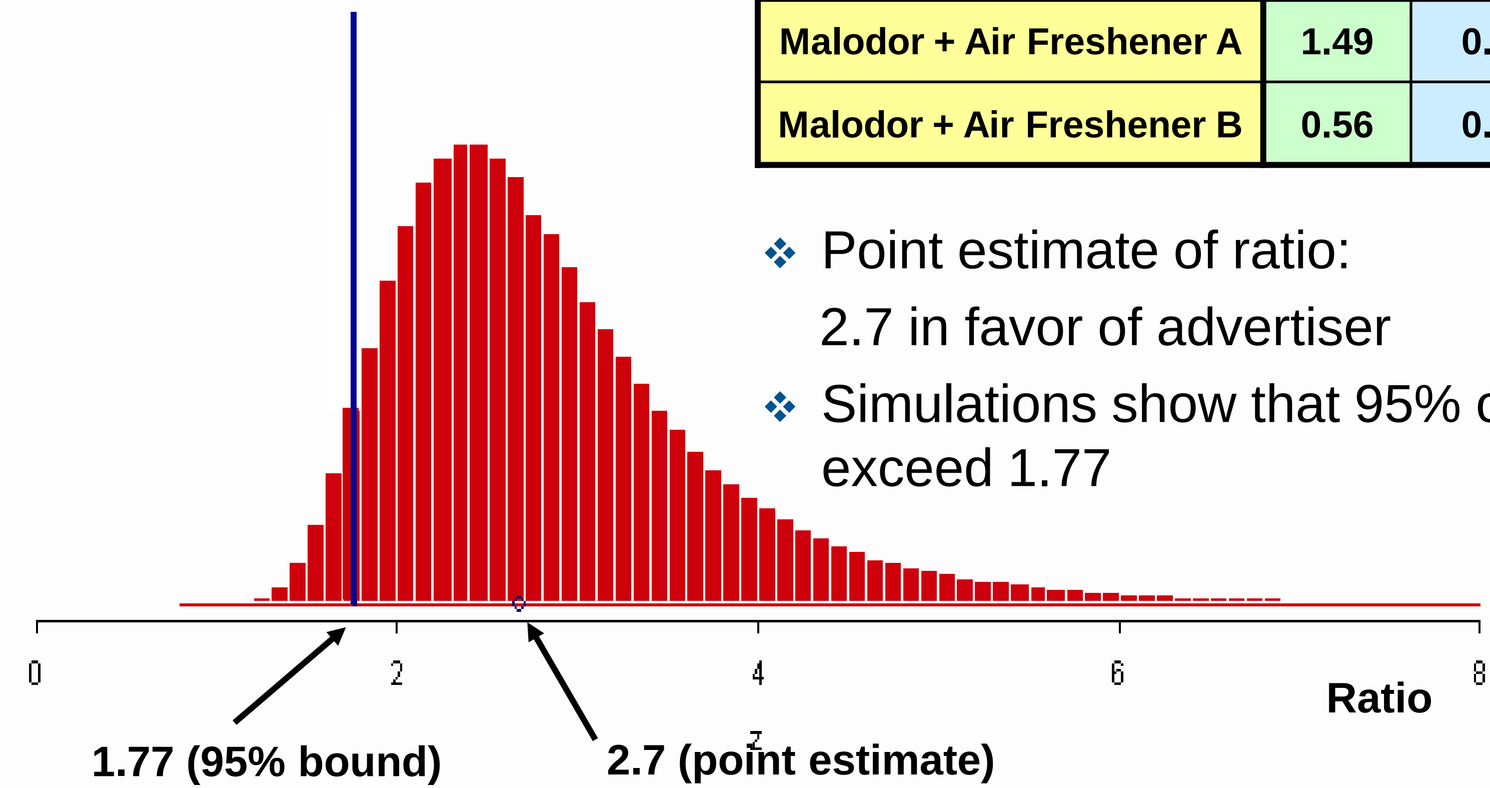
Condition	*d′*	Variance
Malodor + Air Freshener A	1.49	0.027
Malodor + Air Freshener B	0.56	0.022

- Point estimate of ratio: 1.49/0.56 = 2.7

Distribution of Possible Ratios

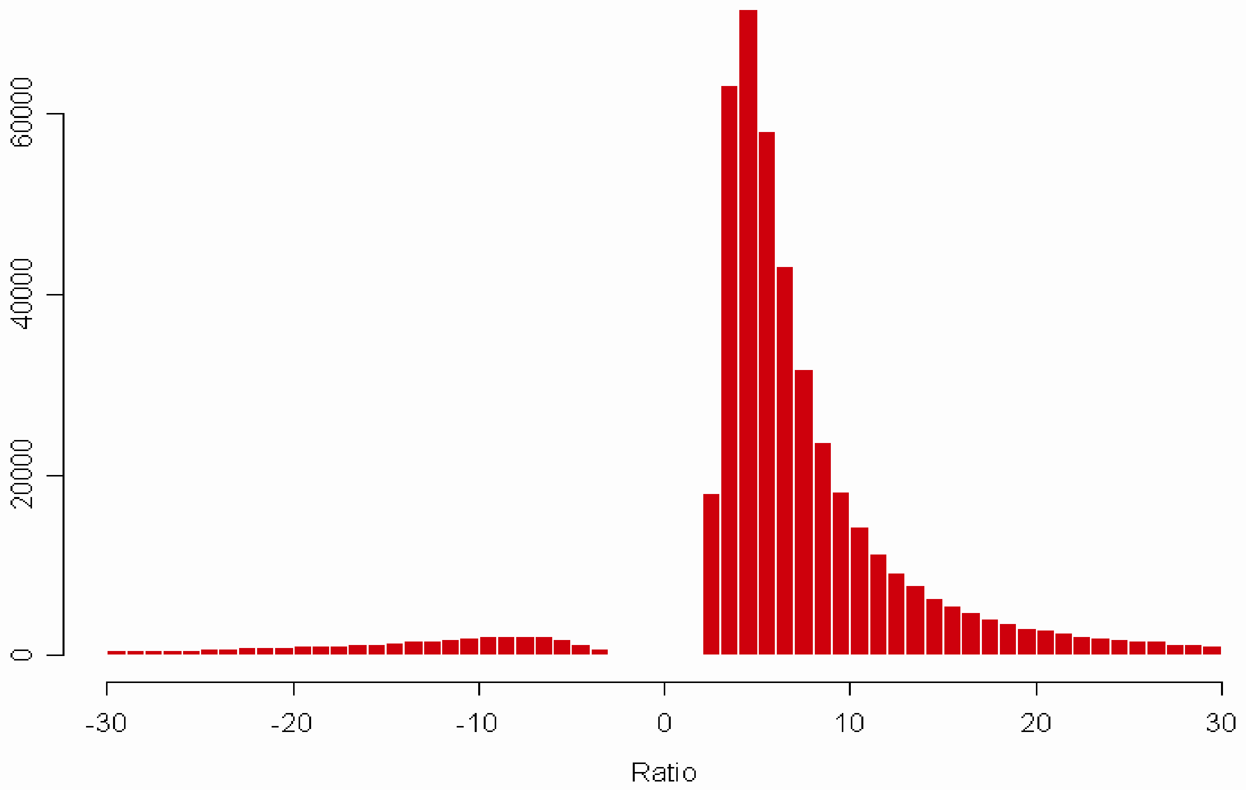
Condition	*d'*	Variance
Malodor + Air Freshener A	1.49	0.027
Malodor + Air Freshener B	0.56	0.022

- Point estimate of ratio:
  2.7 in favor of advertiser
- Simulations show that 95% of ratios exceed 1.77



Finding a Lower Bound in General

- How to determine lower confidence bound for ratios?
- How best to handle the occurrence of negative ratios?



Motivation

- Four cases of malodor treatment
- Two treatments and a malodor control in each case

	Treatment 1			Treatment 2			
Case	**P_c**	***d′***	**Variance of *d′***	**P_c**	***d′***	**Variance of *d′***	**Ratio**
1	0.85	1.47	0.047	0.65	0.54	0.033	2.72
2	0.85	1.47	0.047	0.55	**0.18**	0.032	8.17
3	0.60	0.36	0.032	0.55	**0.18**	0.032	2.00
4	0.55	0.18	0.032	0.52	**0.07**	0.031	2.67

- Want lower 95% confidence bounds in each case

Background – Ratios of Normal RV's

- Geary (1930) - First attempts at pdf
- Fieller (1932)
  - Derived the pdf and cdf of *X*/*Y* under certain restrictions
  - Showed that moments of the ratio do not exist
  - Provided an approximation to the cdf when μ_y/σ_y is large
- Marsaglia (1965) - Discussed alternative forms
- Hinkley (1969) - Evaluated Fieller's approximation
- Ennis et al. (2008)
  - Confidence bounds for positive ratios of normal random variables, *Communications in Statistics*, *Theory and Methods* 37, 307-317

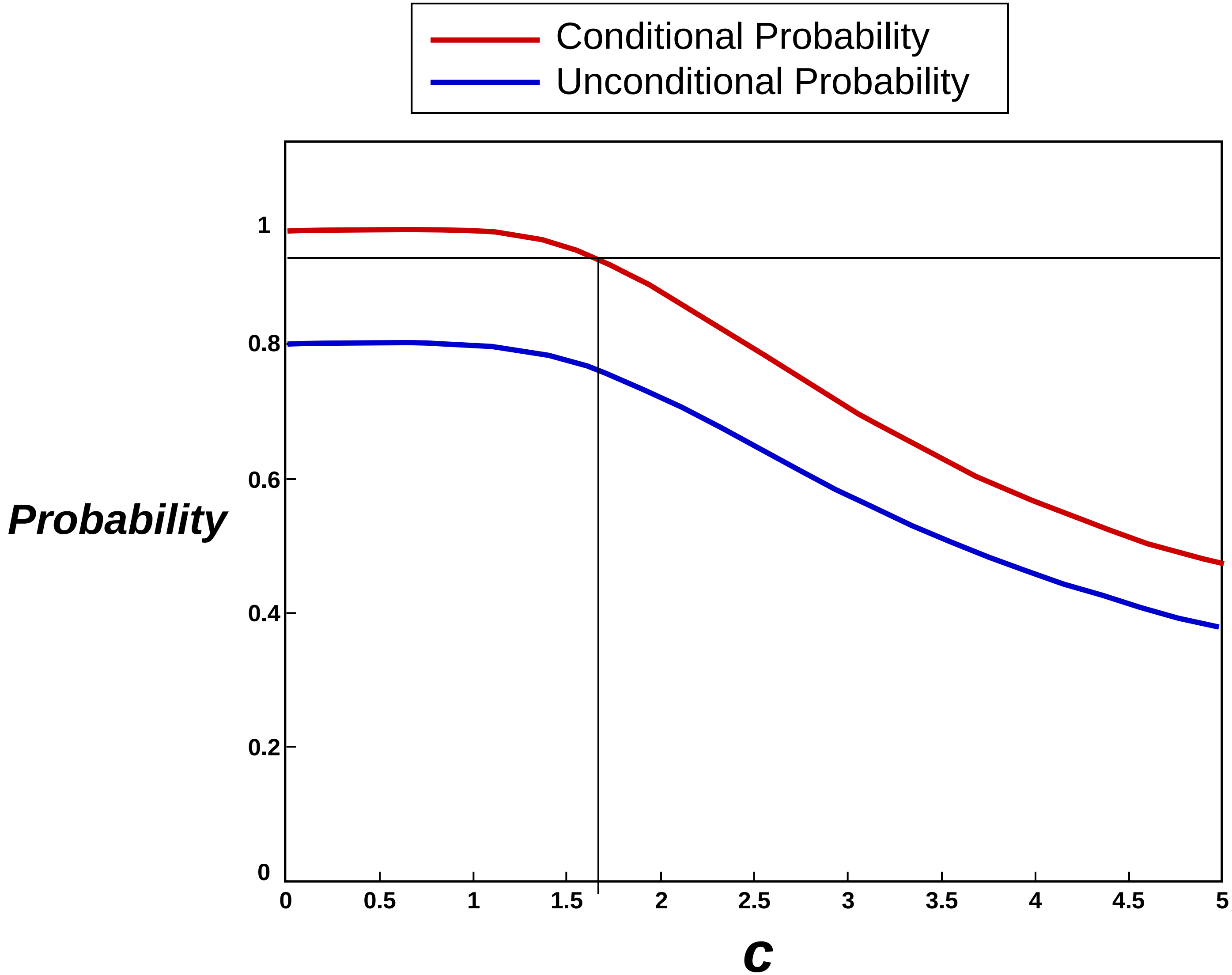
Probability Statements for *Y* > 0

- Suppose (X,Y) is a bivariate normally distributed random variable with mean (μ_X,μ_Y) and covariance matrix

$$\begin{pmatrix} \sigma_x^2 & Cov_{xy} \\ Cov_{xy} & \sigma_y^2 \end{pmatrix}$$

- Suppose c is a positive constant
- Unconditional probability: $\Pr(X/Y > c \text{ and } Y > 0)$
- Conditional probability: $\Pr(X/Y > c \mid Y > 0)$

Conditional vs Unconditional Probability



Conditional Probability

$$P(X/Y > c \mid Y > 0) = \frac{P(X - cY > 0 \text{ and } Y > 0)}{P(Y > 0)}$$

- P(X - cY > 0 and Y > 0) involves an integral over a bivariate normal of the form

$$B(\mu_1, \mu_2; \rho) = \int_0^\infty \int_0^\infty f(\mathbf{x}) d\mathbf{x}$$

where

$$\mu_1 = \frac{\mu_x - c\mu_y}{\sqrt{\sigma_x^2 + c^2\sigma_y^2 - 2cCov_{xy}}} \quad \text{and} \quad \mu_2 = \frac{\mu_y}{\sigma_y}$$

Conditional Probability

- The conditional probability can be reduced to

$$P(X/Y > c \mid Y > 0) = G(\mu_1, \mu_2; c) + \Phi(\mu_1)$$

where *G* is a single integral expression that tends to 0 as μ_y/σ_y goes to infinity

- $G(\mu_1, \mu_2; c) + \Phi(\mu_1)$ is a decreasing function of *c*

- To establish the (1-α)x100% bound, we solve

$$G(\mu_1, \mu_2; c) + \Phi(\mu_1) = 1 - \alpha$$

Fieller as a Special Case

- We have seen that

$$P(X/Y>c\,|\,Y>0)=G(\mu_1,\mu_2;c)+\Phi(\mu_1)$$

- If $\mu_y >> \sigma_y$ then

$$P(X-cY>0\,|\,Y>0)\approx\Phi(\mu_1)$$

- $\Phi(\mu_1)$ is Fieller's approximation (c.f. Hinkley)

Confidence Bounds for Examples

	Treatment 1			Treatment 2				
Case	**P_c**	***d′***	**Variance of *d′***	**P_c**	***d′***	**Variance of *d′***	**Ratio**	**Lower 95% Bound (*c*)**
1	0.85	1.47	0.047	0.65	0.54	0.033	2.72	1.612
2	0.85	1.47	0.047	0.55	0.18	0.032	8.17	**2.851**
3	0.60	0.36	0.032	0.55	0.18	0.032	2.00	**0.257**
4	0.55	0.18	0.032	0.52	0.07	0.031	2.67	**None**

- For case 1, the conditional probability and Fieller’s approximation are nearly identical
- For cases 2, 3 and 4 Fieller’s approximation is not appropriate and should not be used

Summary

- Classical approaches to confidence bounds for ratios of normally distributed random variables are not always applicable to practical cases that arise in product testing
- Using a conditional probability approach, a solution to the problem of determining $1 - \alpha$ confidence bounds for ratios of normal random variables has been derived
- This new method is applicable regardless of the relative sizes of of μ_y and σ_y
- The classical approach based on Fieller's approximation is a special case of the new method

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