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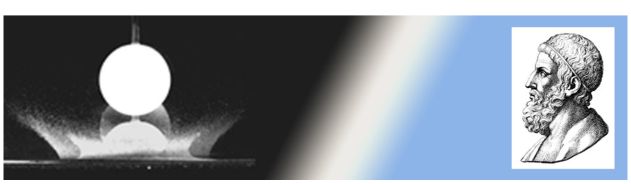

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IFPress® Research Paper 902 (2009) 1-4

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Confidence Bounds for Multiplicative Comparisons

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Abstract

Ratios of normal random variables arise in many applications involving item comparisons. Recent work has extended the classical work in this area and established a method to obtain confidence bounds for the ratio which is conditional on a positive denominator. In this paper the authors offer a new perspective that eliminates the need for such a condition. This perspective involves a clear distinction between ratio and multiplicative ideas and allows for the construction of lower confidence bounds that can be used to support multiplicative statements.

Keywords: Ratios of normal random variables; confidence intervals; estimation; multiplicative comparisons; ratio statements; multiplicative statements

1. Introduction

The statement "Item A is twice as efficacious as item B" is a multiplicative statement. If this statement is made concerning the relative efficacy of two products and this statement appears in advertising in the United States, such a statement can be challenged under the false advertising provisions of the Lanham Act (US Code Chapter 15, section 1125). There are many other applications involving multiplicative statements that arise in a broad range of disciplines and the veracity of such statements needs to be considered from a statistical viewpoint.

Historically it has been common to treat such multiplicative statements as ratio statements. Ratios of normal random variables have long been considered but since the higher moments of the ratio of two independent normal random variables do not exist, classical theory on ratios of normal random variables has been limited to cases in which the mean of the denominator is large relative to its standard deviation (Geary, 1930; Fieller, 1932; Hinkley, 1969). Recently the authors of this present paper used a probabilistic based approach to extend the classical work of Fieller and others to a general setting in which the denominator could be negative (Ennis *et al.*, 2008).

Despite the progress that has been achieved using ratios to assess multiplicative statements, there remain subtle but important shortcomings that lead us to propose a seemingly trivial but surprisingly powerful shift in perspective. Instead of providing improved methods for finding confidence bounds for ratios we demonstrate how one can determine lower confidence bounds for multiplicative statements directly and we show that these lower bounds are superior to those obtained using a ratio based approach.

2. Motivation - Ratio Statements

Suppose that $(X,\ Y)$ is a bivariate normally distributed random variable with mean vector $(\mu_x,\ \mu_y)$ and variance-covariance matrix $\mathrm{W}=\begin{pmatrix} \sigma_x^{\ 2} & Cov_{xy} \\ Cov_{xy} & \sigma_y^{\ 2} \end{pmatrix}$. To evaluate the reasonableness of a statement that X is at least twice as large as Y, it has been common historically to consider how likely it is that X/Y exceeds 2. When one forms the ratio X/Y there are two ways that the ratio can be positive; either both of the terms involved can be positive or both terms can be negative. If the goal is to support a superiority statement about X, for instance that a product X is at least twice as efficacious as a product Y, then one would only wish to consider the case in which X and Y were both positive, as negative ratios are meaningless and positive ratios arising from double negatives are misleading.

To eliminate such troublesome possibilities, Ennis *et al.* considered a conditional probability

$$\Pr(X/Y > c \mid Y > 0). \quad (1)$$

To determine this probability, Ennis *et al.* derived a single integral expression for the unconditional probability $\Pr(X/Y > c \text{ and } Y > 0)$ and used this expression to find confidence bounds for ratio statements based on the conditional probability. In addition Ennis *et al.* showed that this conditional probability converges to the probability arising from the classical work of Fieller as μ_Y/σ_Y tends to infinity. The contribution of Ennis *et al.* was that no prior method existed for producing ratio claims when the denominator had a reasonable chance of returning negative values[1]. Nonetheless the method described in Ennis *et al.* generally leads to conservative lower bounds for the ratio X/Y as the performance of Y is artificially supported by the condition that Y be positive[2].

3. Multiplicative Statements

We now change perspective and consider the statement $X > cY$ instead of the statement $X/Y > c$. Assuming that c is positive, meaningful values for the $X/Y > c$ only occur when both X and Y are positive. Although the statement $X/Y > c$ could also be true for X and Y negative it would be meaningless. On the other hand, the statement $X > cY$ is meaningful as long as X is positive, regardless of the sign of Y. Thus we consider

$$\Pr(X > \mathrm{c}Y \text{ and } X > 0). \quad (2)$$

To compute this probability, we set $\mathbf{Z} = (X - cY,\ X)$. Then $\mathbf{Z}$ is bivariate normal with mean $\boldsymbol{\mu}$ and covariance matrix V, where $\boldsymbol{\mu} = (\mu_x - c\mu_y,\ \mu_x)$ and

$$\mathrm{V} = \begin{pmatrix} \sigma_x^{\ 2}+c^2\sigma_y^{\ 2}-2cCov_{xy} & \sigma_x^{\ 2}-cCov_{xy} \\ \sigma_x^{\ 2}-cCov_{xy} & \sigma_x^{\ 2} \end{pmatrix}.$$

Thus,

$$\Pr(X-\mathrm{c}Y > 0 \text{ and } X > 0) =$$

$$(2\pi)^{-1}\int_0^\infty\!\!\int_0^\infty \frac{\exp\{-0.5(\mathbf{z}-\boldsymbol{\mu})'\,\mathrm{V}^{-1}(\mathbf{z}-\boldsymbol{\mu})\}}{|\mathrm{V}|^{1/2}}d\mathbf{z}. \quad (3)$$

This double integral can be resolved into a single integral using an identity proved in Ennis *et al.* (2008):

$$(2\pi)^{-1}\int_0^\infty\!\!\int_0^\infty \frac{\exp\{(-0.5(\mathbf{z}-\boldsymbol{\mu})'\,\mathrm{V}^{-1}(\mathbf{z}-\boldsymbol{\mu})\}}{|\mathrm{V}|^{1/2}}d\mathbf{z} =$$

$$(2\pi)^{-1}\int_0^\rho (1-t^2)^{-0.5}\exp\{-\frac{(\mu_1^{\ 2}-2t\mu_1\mu_2+\mu_2^{\ 2})}{[2(1-t^2)]}\}dt+\Phi(\mu_1)\Phi(\mu_2), \quad (4)$$

where $\rho = \dfrac{\sigma_x^{\ 2}-cCov_{xy}}{\sqrt{(\sigma_x^{\ 2}+c^2\sigma_y^{\ 2}-2cCov_{xy})\sigma_x^{\ 2}}}$,

$\mu_1 = \dfrac{\mu_x-c\mu_y}{\sqrt{(\sigma_x^{\ 2}+c^2\sigma_y^{\ 2}-2cCov_{xy})}}$, and $\mu_2 = \dfrac{\mu_x}{\sigma_x}$.

Given μ_x, μ_y, σ_x, σ_y and Cov_{xy}, let $F(c)$ be the right hand side of (4). Note that F is a monotonically decreasing function of c and that $F(0) = \Pr(X > 0)$. Thus, for a given α level, we first check that $F(0) > 1-\alpha$ and we then determine a lower $(1-\alpha)$ confidence bound for c by finding the solution of

$$F(c) = (1-\alpha). \quad (5)$$

4. Examples

We now reanalyze Example 1 of Ennis *et al.* and compare the confidence bounds generated there to those arising from the multiplicative method of this present paper. In Ennis *et al.*, there was an interest in comparing the relative efficacy of four sets of malodor treatments in odor chambers designed for such comparisons. Table 1 shows results for independent two-alternative forced choice (2-AFC) experiments in which a choice was made between two alternatives by one hundred assessors in each case. Recall that in the 2-AFC, the task is to choose one of two items that is perceived to be greatest, or least, with respect to some attribute. Results of a 2-AFC experiment are then expressed as a choice probability, P_c, for one of the items. Assuming that the perceptual distributions corresponding to each item are distributed normally with equal variance and that δ is the difference between their means (Thurstone, 1927), then $\delta = \sqrt{2}\,\Phi^{-1}(P_c)$. Other methods based on a similar theoretical foundation can be used to estimate δ (Böckenholt, 1992; Ennis *et al.*, 1988a,b; Hacher and Ratcliff, 1979; Rousseau and Ennis, 2001, 2002). The sample estimate of δ is called d' and, since our examples involve large samples, we

	Treatment 1			Treatment 2				95% Lower Bound	
Case	Proportion	d'	Variance of d'	Proportion	d'	Variance of d'	Ratio	Ratio	Multiplicative
1	0.85	1.47	0.047	0.65	0.54	0.033	2.72	1.612	1.612
2	0.85	1.47	0.047	0.55	0.18	0.032	8.17	2.851	2.951
3	0.60	0.36	0.032	0.55	0.18	0.032	2.00	0.257	0.289
4	0.55	0.18	0.032	0.52	0.07	0.031	2.67	None	None

Table 1. Four cases involving the 2-AFC method in which two treatments are independently compared to a common control.

1. In an advertising claims context, a measure of the efficacy of the competitor's product would appear in the denominator.

2. We will see one large class of applications for which the ratio approach of Ennis *et al.* leads to conservative lower bounds in Section 5.

assume that this estimate is normally distributed. Note that although P_c must fall in the interval [0,1], P_c is mapped to δ which lies in the interval $(-\infty,\infty)$. As an estimate of δ, d' may therefore be positive or negative. The variance of this estimate can be obtained by inverting the second derivative of the likelihood function at d'. The likelihood function for the 2-AFC is

$$L = \binom{n}{r} P_c^{\,r}(1-P_c)^{n-r},$$

where n is the number of trials (2-AFC judgments), r is the number of choices for one of the items and $P_c=\Phi(\delta/\sqrt{2})$. The second derivative of the likelihood function can be determined numerically using a finite difference method.

Returning to our example, we refer to the method of Ennis *et al.* as the ratio approach while referring to the method of this present paper as the multiplicative approach. Table 1 shows the choice proportions and d' values for each case given in Ennis *et al.*, followed by lower 95% confidence bounds for c values obtained by using either the ratio approach or the multiplicative approach. All of the bounds coming from the multiplicative approach are at least as high as those coming from the ratio approach. One can explain this advantage by saying that by using the multiplicative approach an advertiser benefits from the possibly deleterious performance of its rival, an advantage that is lost when one uses the ratio approach.

As a second example we have included Table 2. Table 2 shows lower 95% confidence bounds for c values for a series of experiments similar to those used for Table 1. In each experiment the correct response proportion was 0.64 for Treatment 1 and 0.52 for Treatment 2. Thus the only noteworthy differences between the rows are the sample sizes. From this table we see the greater effectiveness of the multiplicative approach in terms of substantiating comparative statements. In particular note that, for a sample size of 100, the ratio approach yields a lower 95% confidence bound of 0.913 while the multiplicative approach yields a lower 95% confidence bound of 1.081. This means that the multiplicative approach would support a statement such as "8% better" while the ratio approach would not support any superiority statement. Further note that a given lower confidence bound derived from the multiplicative approach requires fewer subjects to attain than would be required by the ratio approach. This second point demonstrates that there could be a cost benefit to using the multiplicative approach.

Sample Size	Ratio Lower Bound	Multiplicative Lower Bound
100	0.913	1.081
125	1.096	1.259
150	1.236	1.394
200	1.474	1.622
300	1.863	1.993
500	2.333	2.436

Table 2. Lower 95% confidence bounds for six cases involving the 2-AFC method. In each case the proportions of correct responses when each treatment is independently compared to the control are 0.64 and 0.52.

5. Theoretical Comparison

In the examples given in the previous section the lower confidence bounds produced by the multiplicative approach were always at least as large as the lower confidence bounds produced by the ratio approach. In this section we show that this relationship holds as long as the random variables are not positively correlated[3]. To see this, suppose that X and Y are not positively correlated and suppose that c is any positive number. We start with

$$P(X>0 \mid Y<0) \geq P(X>0 \mid Y>0) \geq P(X>cY \mid Y>0) \quad (6)$$

and obtain

$$\frac{P(X>0 \textit{ and } Y<0)}{P(Y<0)} \geq \frac{P(X>cY \textit{ and } Y>0)}{P(Y>0)}. \quad (7)$$

From this we see

$$P(X>0 \textit{ and } Y<0) \geq \frac{(1-P(Y>0))}{P(Y>0)} P(X>cY \textit{ and } Y>0), \quad (8)$$

which gives

$$P(X{>}0 \textit{ and } Y{<}0) + P(X{>}cY \textit{ and } Y{>}0) \geq \frac{P(X{>}cY \textit{ and } Y{>}0)}{P(Y{>}0)} \quad (9)$$

The right hand side of this inequality is the conditional probability. On the other hand, the two terms on the left hand side together comprise $P(X>cY \textit{ and } X>0)$. Thus

$$P(X>cY \textit{ and } X>0) \geq P(X>cY \mid Y>0). \quad (10)$$

From this inequality we infer that whenever the ratio approach yields a positive lower confidence bound then the multiplicative approach will yield a positive lower confidence bound that is at least as large.

6. Conclusion

In this paper we have provided the machinery to produce lower confidence bounds for multiplicative statements by way of a seemingly trivial yet surprisingly powerful change of perspective. We have provided examples illustrating the sometimes sizable differences between the lower confidence bounds produced by a ratio approach and the lower confidence bounds produced by this new multiplicative approach and we have demonstrated that when the random variables involved are not positively correlated the multiplicative approach always yields higher bounds.

3. If the random variables are positively correlated then equation (6) will not hold for small values of c.

7. Acknowledgement

We thank Dr. William Raynor for his suggestion that we consider the presence of negative ratios due to negative denominators in the determination of confidence bounds.

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